Evil Puzzles set by Prison Wardens

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The Chessboard Puzzle

The warden will arrange coins on a chessboard with one prisoner watching and then point to a particular coin. The prisoner must then turn over a single coin on the board and leave.

The other prisoner then is let in and must identify the coin the warden pointed at.

What strategy can the prisoners use?

As usual failure results in death for both and success means freedom!

Hints

- It is always possible to succeed with the given puzzle
- There must be a square where flipping the coin does not affect the square chosen.
- Success is only certain when the number of squares is a power of two.
- Try with just two squares. Then four.
- (A xor B) xor A = B

- Earliest I could find was 2009 but they were not the originator

Many ways of solving

But they all come to the same thing.

Can think of all the squares having a binary address. Or of them all being in a 2x2x..2 hypercube. Or of them all being in a binary tree.

Using a binary address the 64 chessboard squares can be addressed as 000000, 000001, 000010, ..., 111110, 111111.

Ensure that exclusive orring all the ones that are heads gives the address of the one pointed to by flipping the one given by the address got by exclusive orring all the heads ones with the target address.

Four squares are probably quite enough to check this out!

Proof that it can't be done with 3 coins

Coins before	After a coin is turned 1st	2nd	3rd
HHH	THH	HTH	HHT
HHT	THT	HTH	HHH
HTH	TTH	HHH	HTT
HTT	TTT	HHT	HTH
THH	HHH	TTH	THT
THT	HHT	TTT	THH
TTH	HTH	THH	TTT
TTT	HTT	THT	TTH

Each of the three after positions must correspond to a different place the warden pointed at call them A, B and C - 1 2 3 just as good but more obvious.

From the first line HHH say
THH -> A, HTH -> B, and HHT -> C

From the line for HTT and the assignment for HHH
HHT -> C and HTH -> B therefore TTT -> A

Then from the line for TTH
HTH -> B, THH -> A and TTT -> A

So there's no way to say the warden pointed at C!

100 Prisoners Puzzle

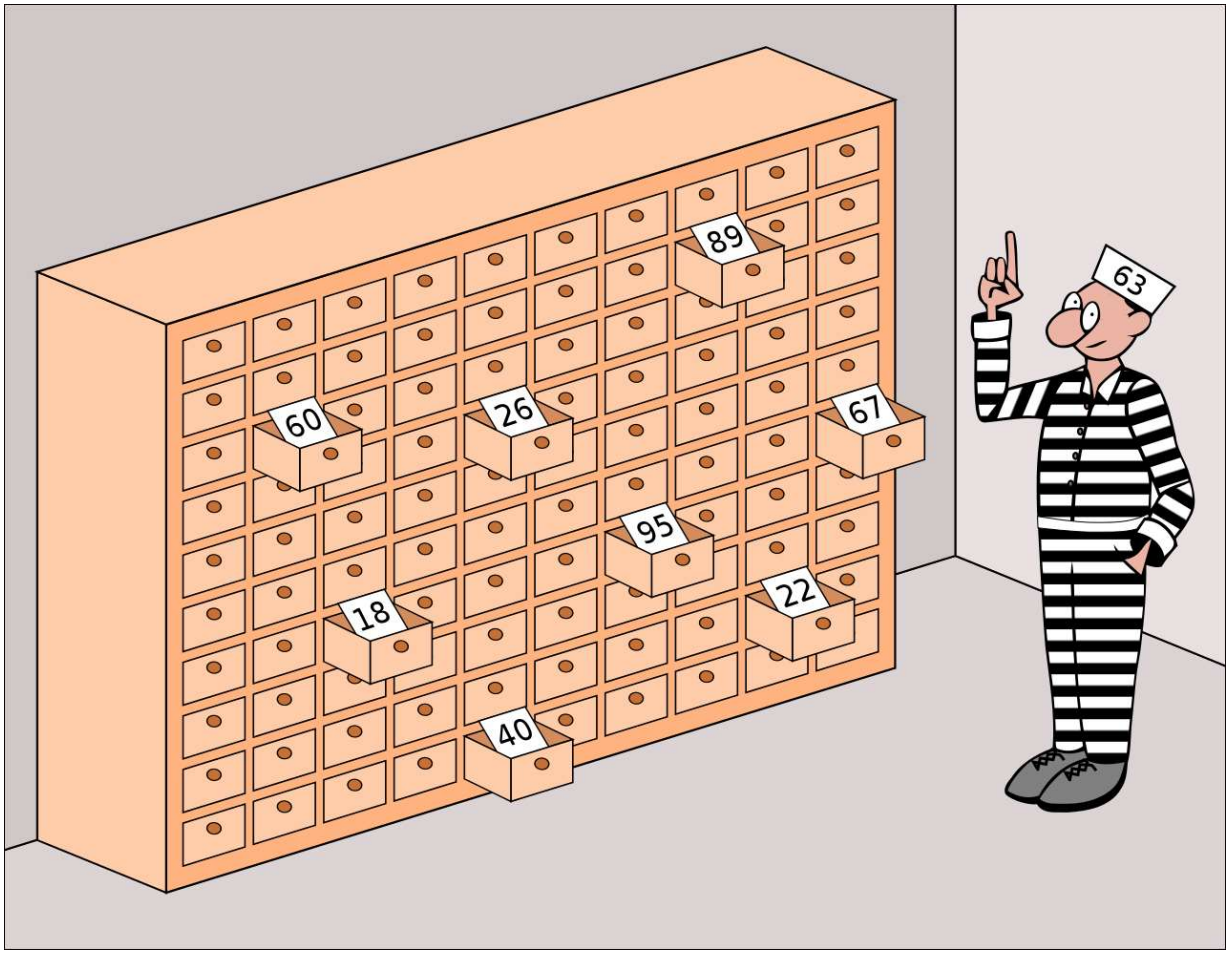
The warden offers 100 death row prisoners a chance. He randomly puts one of their names into each of their 100 lockers.

They each get a chance to open fifty lockers. If they all find their names their lives will be spared, else as usual all will die. They can’t move the names.

The lockers are closed again after each try and the prisoner put in a different block so they can’t communicate back.

What is their best strategy? They’re allowed to confer beforehand.

You can use numbers if you prefer!

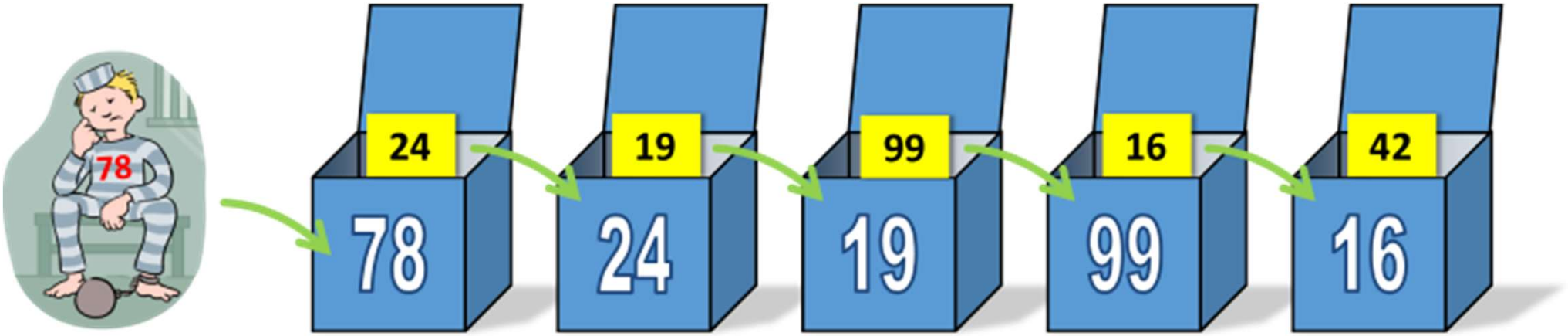


Hints

- This time they are not assured of success
- With no strategy then probability of each person finding their name is ½. The probability everyone will find their own name is $1/2^{100}$, which is ≈ $8/10^{31}$. Or basically they're toast.
- They'll find another name in each locker they open.
- Their own name is the only one that is special to them!

- In Wikipedia as the 100 prisoners problem.
  Proposed by Dutch computer scientist Peter Bro Miltersen in 2003

The strategy



The chain will eventually form a circle. If the length is fifty or less then you have succeeded. If any chain is more than 50 you are all kaput.

Image from
https://datagenetics.com/blog/december42014/index.html

So each prisoner has a better than ½ chance?

No. The strategy of following the circle does not improve the chances of each prisoner from ½.

Since the name in each locker opened is random there is no improvement in the prisoners chance over just picking fifty lockers at random.

Where the strategy gives an advantage is that it ties the chances of the prisoners together. If any one in the largest cycle fails they all fail. If any one succeeds they all succeed.

So the chance of them all succeeding is the probability that the largest cycle is 50 or smaller.

The chance of success

The chance of success is 1 – chance the largest cycle is 51 or more.

At most one cycle of length greater than 50.

For a cycle length n there are $\binom{100}{n}$ ways to select the numbers and the numbers can be arranged in $(n-1)!$ ways where it doesn't matter where the start is. The remaining numbers can be arranged in $(100-n)!$ ways.

So total number of permutations of 1 .. 100 with a cycle of length n is

$$\binom{100}{n}(n-1)(100-n)! = \frac{100!}{n}$$

.. continued

The total number of permutations is $100!$

So the chances of success are

$$1 - \frac{1}{100!}\left(\frac{100!}{51} + \frac{100!}{52} + \cdots + \frac{100!}{100}\right) \approx 0.31183$$

In the limit as the number of prisoners increases this tends to

$$1 - \ln 2 \approx 0.30685$$

Using $\sum_{k=1}^{N} \frac{1}{k} = H_N \approx \ln N + \gamma$

So their chances are always greater than 0.3 !

The cat-burglar makes a big difference

A cat burglar manages to get into the room beforehand. Their inside security isn't as great as external!

He is able to inspect all the lockers, oh dear there's a cycle of more than 50! He can't remember everything and he doesn't want to disturb anything or the warden might notice.

What's the minimum information he should bring back?

Hint – you want to split the cycle of more than 50 into two.

Cut the longest cycle in two

He should bring back the names of two people who are as far apart as possible in the longest cycle.

If a person reaches one of the names he should go to the locker of the other one. This will split the cycle into two cycles of length less than or equal to 50.

The Warden is totally evil

The warden has a bug in the room where the prisoners discuss their strategy. With a malicious grin he goes off to arrange the contents of the lockers to ensure their plans fail. The largest cycle will be more than 50! Haw haw haw he guffaws.

However one of the guards has a heart and sees the warden slink off. He knows of the warden's ultra evil ways. He secretly warns the prisoners. They have to change their plans at the last moment!

What should they do?

Hints

- As for the cat burglar scenario, the locker got at by a name doesn't have to be the one with the name on!
- Need everyone to get into a cycle where their name occurs in a locker.
- The first locker chosen must be one that would be got at if the previous locker in the chain had the required name.
- Need a one to one mapping between the names and lockers.
- Any random mapping will do! If pressed for time they could go to the locker with the name and then count on 13 for example which will probably defeat the evil wardens plans.

The prisoners are free!

The prisoners are free! They tell of their experiences and after a long inquiry the warden becomes a prisoner in his own jail!

“Perhaps I should have stuck to putting rats into mazes”, he says, “But they are all vermin and deserve it”.